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不可压渗流驱动问题混合有限元/间断有限元 耦合格式的误差分析*

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摘 要: 多孔介质中渗流驱动问题与环境污染和油藏开采等问题密切相关, 是当今的研究热点. 对具有分子扩散和弥散效应的不可压渗流驱动问题, 本文用混合有限元/间断有限元耦合格式来求解, 即用混合有限元方法求解压力方程, 用对称内罚间断有限元方法逼近浓度方程. 运用比剪切算子更为便捷的归纳假设和插值投影, 导出了先验 h_p 误差估计.

关键词: 混合有限元; 间断有限元; 渗流驱动

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1 引言

我们考虑下列不可压缩的多孔介质渗流驱动问题

$$\left\{ \begin{array}{ll} \nabla \cdot \mathbf{u} = -\nabla \cdot (a(c)\nabla p) = q, & (x, t) \in \Omega \times J, \\ \phi \frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c - \nabla \cdot (D(\mathbf{u})\nabla c) = (\hat{c} - c)q, & (x, t) \in \Omega \times J, \\ \mathbf{u} \cdot \mathbf{n} = 0, & (x, t) \in \partial\Omega \times J, \\ D(\mathbf{u})\nabla c \cdot \mathbf{n} = 0, & (x, t) \in \partial\Omega \times J, \\ c(x, 0) = c_0(x), & x \in \Omega, \end{array} \right. \quad (1)$$

其中 Ω 是二维的有界多边形区域, $\partial\Omega$ 是其边界, $J = (0, T]$, \mathbf{n} 表示 $\partial\Omega$ 的单位外法向量; $\mathbf{u}(x, t)$ 为流体的 Darcy 速度, $p(x, t)$ 为流体的压力, $c(x, t)$ 为注入流体的浓度, $\phi(x) > 0$ 为有效介质孔隙度, $D(\mathbf{u})$ 表示扩散/弥散张量.

$$D(\mathbf{u}) = d_m \mathbf{I} + |\mathbf{u}|(\alpha_l \mathbf{E}(\mathbf{u}) + \alpha_t (\mathbf{I} - \mathbf{E}(\mathbf{u}))),$$

其中 $\mathbf{E}(\mathbf{u})$ 是投影到 \mathbf{u} 方向上的张量, 其 (i, j) 分量为

$$(\mathbf{E}(\mathbf{u}))_{i,j} = \frac{u_i u_j}{|\mathbf{u}|^2},$$

$d_m > 0$ 是分子扩散系数; $\alpha_l \geq 0$ 和 $\alpha_t \geq 0$ 分别为纵向和横向的弥散系数. q 为井内的流量, $q = q^+ + q^-$, 其中 $q^+ = \max(q, 0)$, $q^- = \min(q, 0)$. 假定 q 和 $\frac{\partial q}{\partial t}$ 都是有界的, \hat{c} 表示注入

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井处注入的浓度 ($q > 0$) 和产出井处剩余流体的浓度 ($q < 0$), 系数 $a(c) > 0$ 有界, $\frac{\partial a(c)}{\partial c}$ 是一致有界且关于 c 是 Lipschitz 连续的.

混合有限元方法能够使压力和 Darcy 速度取得相同的收敛阶, 被广泛应用于多孔介质问题的数值模拟^[1,2]. 间断有限元方法是一类非协调元^[3,4], 因为它有着很好的物理和数值特性, 在数值模拟中很受欢迎. 文献 [5,6] 对于只有分子扩散效应的可压渗流驱动问题, 采用了混合有限元/间断有限元耦合格式. 在分析中, 文献 [5] 引入了剪切算子, 而文献 [6] 运用了归纳假设^[2]. 不过他们的分析仅只限于没有弥散的特殊情况, 实际上分子扩散和弥散的影响都是存在的, 这给分析带来了一定的困难. 文献 [7] 中, 对同时具有分子扩散和弥散效应的不可压渗流驱动问题, 利用剪切算子导出了半离散的混合有限元/间断有限元方法的耦合格式的误差估计. 本文研究了同时具有分子扩散和弥散效应的不可压渗流驱动问题混合有限元/间断有限元耦合格式, 并采用归纳假设进行误差分析. 这样避免了象文献 [5,7] 那样选择剪切算子中正常数的麻烦, 而且归纳假设可以直接由离散格式和已得误差估计式证得.

2 混合有限元/间断有限元耦合格式

设 T_h 是 Ω 的拟一致网格剖分 (三角形或四边形). Γ_h 为所有内部边的集合, $h = \max_{E \in T_h} h_E$ 为最大单元尺寸. 对 $s \geq 0$, 定义下列间断 Sobolev 空间

$$H^s(T_h) = \{v \in L^2(\Omega) : v|_E \in H^s(E), E \in T_h\}.$$

设 $E_i \in T_h$, $E_j \in T_h$, $\gamma = \partial E_i \cap \partial E_j \in \Gamma_h$ (其外法向 \mathbf{n} 指向 E_i 的外部). 定义 $v \in H^s(T_h)$, $s > 1/2$ 的平均和跳跃为

$$\{v\} = \frac{1}{2}((v|_{E_i})|_\gamma + (v|_{E_j})|_\gamma), \quad [v] = (v|_{E_i})|_\gamma - (v|_{E_j})|_\gamma.$$

间断有限元空间定义为

$$D_r(T_h) = \{v \in L^2(\Omega) : v|_E \in P_r(E), E \in T_h\},$$

其中 $P_r(E)$ 表示在 E 上总的度小于等于 r 的多项式空间. 定义空间

$$\begin{aligned} V &= H(\text{div}; \Omega) = \{\mathbf{u} \in (L^2(\Omega))^2, \text{div} \mathbf{u} \in L^2(\Omega)\}, \\ V^0 &= H_0(\text{div}; \Omega) = \{\mathbf{u} \in H(\text{div}; \Omega), \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0\}, \\ W &= L^2(\Omega). \end{aligned}$$

令 $V \times W$ 关于剖分 T_h 的逼近子空间 $V_k(T_h) \times W_k(T_h)$ 为 k ($k \geq 0$) 阶 Raviart-Thomas 空间 (RT_k). 相应地 V^0 的子空间为 $V_k^0(T_h) = V_k(T_h) \cap V^0$.

本文采用通常的 Sobolev 内积 (\cdot, \cdot) 和范数 $\|\cdot\|_{m,\Omega}$. 特别地, $\|\cdot\|_{0,\Omega}$ 记作 $\|\cdot\|$. 为简单起见, 在积分式 $\int \cdot dx$ 和 $\int \cdot dt$ 中省去了 dx 和 dt , 即分别用 $\int_g \cdot$ ($g = E, \gamma, \Omega$) 和 $\int_0^t \cdot$ 表示关于空间变量的积分和时间变量积分. K, K_i ($i \in N$) 表示一类与 h, r 和 k 无关但可能与问题的解相关的正常数. 这些常数在不同的场合下取不同的值. 用 ε 表示一个可以任意小的正常数.

对任意的 $\psi \in D_r(\mathcal{T}_h)$, 定义双线性形式 $B(\mathbf{u}; c, \psi)$ 和线性泛函 $L(c, \psi)$:

$$\begin{aligned} B(\mathbf{u}; c, \psi) &= \sum_{E \in \mathcal{T}_h} \int_E \mathbf{D}(\mathbf{u}) \nabla c \cdot \nabla \psi - \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{\mathbf{D}(\mathbf{u}) \nabla c \cdot \mathbf{n}\} [\psi] \\ &\quad - \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{\mathbf{D}(\mathbf{u}) \nabla \psi \cdot \mathbf{n}\} [c] + \sum_{E \in \mathcal{T}_h} \int_E (\mathbf{u} \cdot \nabla c) \psi + J_0^\sigma(c, \psi), \\ L(c, \psi) &= \int_{\Omega} (\hat{c} - c) q \psi, \end{aligned}$$

其中 h_γ 表示 γ 的尺寸,

$$J_0^\sigma(c, \psi) = \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_\gamma}{h_\gamma} \int_{\gamma} [c] [\psi]$$

为罚项. $\sigma > 0$ 为罚参数, 它在边 γ 上取值为 σ_γ , 且 $\sigma_* \leq \sigma_\gamma \leq \sigma^*$.

对压力方程, 我们采用混合有限元方法求解. 对浓度方程, 我们采用对称内罚间断有限元方法逼近. 那么问题 (1) 的混合有限元/间断有限元耦合格式为: 寻找 $\mathbf{U} \in L^\infty(J; V_k^0(\mathcal{T}_h))$, $P \in L^\infty(J; W_k(\mathcal{T}_h))$, $C \in L^\infty(J; D_r(\mathcal{T}_h))$, 使得

$$(\nabla \cdot \mathbf{U}, w) = (q, w), \quad \forall w \in W_k(\mathcal{T}_h), \quad (2)$$

$$(\alpha(C) \mathbf{U}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, P) = 0, \quad \forall \mathbf{v} \in V_k^0(\mathcal{T}_h), \quad (3)$$

$$\left(\phi \frac{\partial C}{\partial t}, \psi \right) + B(\mathbf{U}; C, \psi) = L(C, \psi), \quad \forall \psi \in D_r(\mathcal{T}_h), \quad (4)$$

$$(C, \psi) = (c_0, \psi), \quad \forall \psi \in D_r(\mathcal{T}_h), \quad t = 0, \quad (5)$$

其中 $\alpha(c) = 1/a(c)$.

3 插值投影和归纳假设

设 $\tilde{\mathbf{u}}$ 和 \tilde{p} 分别为 \mathbf{u} 和 p 的投影, 满足

$$(\nabla \cdot \tilde{\mathbf{u}}, w) = (q, w), \quad \forall w \in W_k(\mathcal{T}_h), \quad (6)$$

$$(\alpha(C) \tilde{\mathbf{u}}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, \tilde{p}) = 0, \quad \forall \mathbf{v} \in V_k^0(\mathcal{T}_h). \quad (7)$$

记 $\boldsymbol{\rho} = \mathbf{u} - \tilde{\mathbf{u}}$, $\boldsymbol{\sigma} = \tilde{\mathbf{u}} - \mathbf{U}$, $\eta = p - \tilde{p}$, $\pi = \tilde{p} - P$. 假定 $\boldsymbol{\sigma}(\cdot, 0) = 0$, 由文献 [2,8] 可以得到如下投影误差估计

$$\|\boldsymbol{\rho}\| + \|\eta\| \leq K \sum_{E \in \mathcal{T}_h} \frac{h_E^{\min(k+1, \omega_E-1)}}{k^{\omega_E-1/2}} \|p\|_{\omega_E, E}, \quad \left\| \frac{\partial \boldsymbol{\rho}}{\partial t} \right\| \leq K \sum_{E \in \mathcal{T}_h} \frac{h_E^{\min(k+1, \omega_E-1)}}{k^{\omega_E-1/2}} \|p\|_{\omega_E, E},$$

其中 k 为 RT_k 空间的阶数, ω_E 为 p 的正则阶.

根据文献 [9,10], 对 $E \in \mathcal{T}_h$, $w \in H^s(\mathcal{T}_h)$, 存在与 E 和 s 有关但与 c , r 和 h_E 无关的常数 K , 以及 $\tilde{w} \in P_r(E)$, 使得当 $0 \leq q \leq s$ 和 $\mu = \min(r+1, s)$ 时, 有

$$\|w - \tilde{w}\|_{q, E} \leq K \frac{h_E^{\mu-q}}{r^{s-q}} \|w\|_{s, E}, \quad s \geq 0, \quad (8)$$

$$\|w - \tilde{w}\|_{\delta, \partial E} \leq K \frac{h_E^{\mu-\delta-1/2}}{r^{s-\delta-1/2}} \|w\|_{s, E}, \quad s > \frac{1}{2} + \delta, \quad \delta = 0, 1. \quad (9)$$

设 \tilde{c} 为 c 的插值, $c - \tilde{c}$ 和 $\partial(c - \tilde{c})/\partial t$ 满足上述 hp 逼近性质. 令 $\zeta = c - \tilde{c}$, $\xi = \tilde{c} - C$, 并取 $\xi(\cdot, 0) = 0$, 则 $c - C = \xi + \zeta$.

在分析过程中, 需要用到下列归纳假设 (将在第5节进行证明)

$$\|\sigma\|_{L^\infty(J; L^\infty(\Omega))} \leq L, \quad (10)$$

$$\left\| \frac{\partial \sigma}{\partial t} \right\|_{L^2(J; L^\infty(\Omega))} \leq L, \quad (11)$$

其中 L 为某个正常数.

4 误差分析

从式 (6)-(7) 分别减去式 (2)-(3), 得

$$(\nabla \cdot \sigma, w) = 0, \quad \forall w \in W_k(\mathcal{T}_h), \quad (12)$$

$$(\alpha(C)\sigma, v) - (\nabla \cdot v, \pi) = ((\alpha(C) - \alpha(c))\tilde{u}, v), \quad \forall v \in V_k^0(\mathcal{T}_h). \quad (13)$$

取 $w = \pi$, $v = \sigma$, 则

$$(\alpha(C)\sigma, \sigma) - ((\alpha(C) - \alpha(c))\tilde{u}, \sigma) = 0.$$

应用 Hölder 不等式推出

$$|(\alpha(C) - \alpha(c))\tilde{u}, \sigma| \leq K\|\zeta + \xi\| \cdot \|\sigma\|.$$

注意到 $\alpha(C)$ 是有界的, 于是, 得到了压力方程的误差估计

$$\|\sigma\|^2 \leq K(\|\zeta\|^2 + \|\xi\|^2). \quad (14)$$

设 (p, u, c) 是问题 (1) 的解, 它满足

$$\left(\phi \frac{\partial c}{\partial t}, \psi\right) + B(u; c, \psi) = L(c, \psi), \quad \forall \psi \in D_r(\mathcal{T}_h), \quad t \in J. \quad (15)$$

注意到^[2]

$$(\hat{c} - c) - (\hat{C} - C) = \begin{cases} -(\zeta + \xi), & \text{如果 } q > 0, \\ 0, & \text{如果 } q < 0. \end{cases}$$

由于

$$u \cdot \nabla c - U \cdot \nabla C = (\rho + \sigma) \cdot \nabla c + U \cdot \nabla(\zeta + \xi),$$

用式 (15) 减去式 (4), 并取 $\psi = \frac{\partial \xi}{\partial t}$, 可得

$$\begin{aligned} & \left(\phi \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t}\right) + \sum_{E \in \mathcal{T}_h} \int_E D(U) \nabla \xi \cdot \nabla \frac{\partial \xi}{\partial t} + \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} q^+ \xi^2 + J_0^\sigma(\xi, \xi) \right) \\ &= - \left(\phi \frac{\partial \zeta}{\partial t}, \frac{\partial \xi}{\partial t}\right) + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{ (D(u) - D(U)) \nabla c \cdot n \} \left[\frac{\partial \xi}{\partial t} \right] - \sum_{E \in \mathcal{T}_h} \int_E (D(u) - D(U)) \nabla c \cdot \nabla \frac{\partial \xi}{\partial t} \\ &+ \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{ (D(u) - D(U)) \nabla \frac{\partial \xi}{\partial t} \cdot n \} [c] - \sum_{E \in \mathcal{T}_h} \int_E (\rho + \sigma) \cdot \nabla c \frac{\partial \xi}{\partial t} - \sum_{E \in \mathcal{T}_h} \int_E U \cdot \nabla(\zeta + \xi) \frac{\partial \xi}{\partial t} \end{aligned}$$

$$\begin{aligned}
& - \sum_{E \in \mathcal{T}_h} \int_E \mathbf{D}(\mathbf{U}) \nabla \zeta \cdot \nabla \frac{\partial \xi}{\partial t} + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{ \mathbf{D}(\mathbf{U}) \nabla (\zeta + \xi) \cdot \mathbf{n} \} \left[\frac{\partial \xi}{\partial t} \right] - J_0^\sigma \left(\zeta, \frac{\partial \xi}{\partial t} \right) \\
& + \sum_{\gamma \in \Gamma_h} \int_{\gamma} \left\{ \mathbf{D}(\mathbf{U}) \nabla \frac{\partial \xi}{\partial t} \cdot \mathbf{n} \right\} [\zeta + \xi] - \int_{\Omega} \zeta q^+ \frac{\partial \xi}{\partial t} + \frac{1}{2} \int_{\Omega} \frac{\partial q^+}{\partial t} \xi^2 \\
& \equiv \sum_{i=1}^{12} T_i, \quad \forall \psi \in D_r(\mathcal{T}_h), \quad t \in J.
\end{aligned} \tag{16}$$

对项 T_i ($i = 1, 2, \dots, 12$) 进行估计, 我们需要用到下列不等式. 注意到, 当 h 足够小时

$$\begin{aligned}
\|\mathbf{U}\|_{\infty, \Omega} & \leq \|\tilde{\mathbf{u}}\|_{\infty, \Omega} + \|\boldsymbol{\sigma}\|_{\infty, \Omega} \leq M, \\
\left\| \frac{\partial \mathbf{U}}{\partial t} \right\|_{L^2(J; L^\infty(\Omega))} & \leq \left\| \frac{\partial \tilde{\mathbf{u}}}{\partial t} \right\|_{L^2(J; L^\infty(\Omega))} + \left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|_{L^2(J; L^\infty(\Omega))} \leq M,
\end{aligned}$$

其中用到了归纳假设 (10)-(11)、 $\|\tilde{\mathbf{u}}\|_{\infty, \Omega}$ 和 $\left\| \frac{\partial \tilde{\mathbf{u}}}{\partial t} \right\|_{L^2(J; L^\infty(\Omega))}$ 的有界性^[2].

因为 $[c] = 0$, 所以 $T_4 = 0$. 利用 Cauchy-Schwartz 不等式, 可得

$$\begin{aligned}
|T_1| & \leq \varepsilon \left\| \frac{\partial \xi}{\partial t} \right\|^2 + K \left\| \frac{\partial \zeta}{\partial t} \right\|^2, \quad |T_5| \leq \varepsilon \left\| \frac{\partial \xi}{\partial t} \right\|^2 + K(\|\boldsymbol{\rho}\|^2 + \|\boldsymbol{\sigma}\|^2), \\
|T_6| & \leq \varepsilon \left\| \frac{\partial \xi}{\partial t} \right\|^2 + K(\|\nabla \zeta\|^2 + \|\nabla \xi\|^2), \quad |T_{11}| \leq \varepsilon \left\| \frac{\partial \xi}{\partial t} \right\|^2 + K\|\zeta\|^2, \quad |T_{12}| \leq K\|\xi\|^2.
\end{aligned}$$

因为

$$|\mathbf{D}(\mathbf{U})| \leq d_m + \max(\alpha_i, \alpha_t) |\mathbf{U}| \leq M, \quad \left| \frac{\partial \mathbf{D}(\mathbf{U})}{\partial t} \right| \leq K \|\mathbf{U}\|_{\infty, \Omega} \left\| \frac{\partial \mathbf{U}}{\partial t} \right\|_{\infty, \Omega} \leq M.$$

注意到 $\xi(x, 0) = 0$, 对 T_7 关于 t 积分并用分部积分法, 推得

$$\begin{aligned}
\left| \int_0^t T_7 \right| & \leq \left| \sum_{E \in \mathcal{T}_h} \int_E (\mathbf{D}(\mathbf{U}) \nabla \zeta \cdot \nabla \xi)(\cdot, t) \right| + \left| \int_0^t \sum_{E \in \mathcal{T}_h} \int_E \mathbf{D}(\mathbf{U}) \nabla \xi \cdot \nabla \frac{\partial \zeta}{\partial t} \right| \\
& \quad + \left| \int_0^t \sum_{E \in \mathcal{T}_h} \int_E \frac{\partial \mathbf{D}(\mathbf{U})}{\partial t} \nabla \xi \cdot \nabla \zeta \right| \\
& \leq \varepsilon \sum_{E \in \mathcal{T}_h} \int_E |\nabla \xi(\cdot, t)|^2 + K \sum_{E \in \mathcal{T}_h} \int_E |\nabla \zeta(\cdot, t)|^2 \\
& \quad + K \int_0^t \sum_{E \in \mathcal{T}_h} \int_E \left(|\nabla \xi|^2 + |\zeta|^2 + \left| \nabla \frac{\partial \zeta}{\partial t} \right|^2 \right).
\end{aligned}$$

对于 T_8 和 T_{10} , 利用分部积分、Cauchy-Schwartz 不等式、 σ_γ^{-1} 的有界性以及文献 [6] 中的迹不等式引理 3.1 和逆不等式引理 3.2, 可知

$$\begin{aligned}
\left| \int_0^t (T_8 + T_{10}) \right| & \leq \left| \int_0^t \sum_{\gamma \in \Gamma_h} \int_{\gamma} \{ \mathbf{D}(\mathbf{U}) \nabla (\zeta + \xi) \cdot \mathbf{n} \} \left[\frac{\partial \xi}{\partial t} \right] \right| + \left| \int_0^t \sum_{\gamma \in \Gamma_h} \int_{\gamma} \left\{ \mathbf{D}(\mathbf{U}) \nabla \frac{\partial \xi}{\partial t} \cdot \mathbf{n} \right\} [\zeta + \xi] \right| \\
& \leq K \sum_{\gamma \in \Gamma_h} r^{-2} h_\gamma \|\nabla \zeta(\cdot, t) \cdot \mathbf{n}\|_\gamma^2 + \varepsilon J_0^\sigma(\xi(\cdot, t), \xi(\cdot, t)) + K \int_0^t \sum_{\gamma \in \Gamma_h} r^{-2} h_\gamma \left\| \nabla \frac{\partial \zeta}{\partial t} \cdot \mathbf{n} \right\|_\gamma^2
\end{aligned}$$

$$\begin{aligned}
& +K \int_0^t J_0^\sigma(\xi, \xi) + K \int_0^t \sum_{\gamma \in \Gamma_h} r^{-2} h_\gamma \|\nabla \zeta \cdot \mathbf{n}\|_\gamma^2 + K_1 \left(\min_{\gamma \in \Gamma_h} \sigma_\gamma \right)^{-1} \sum_{E \in \mathcal{T}_h} \int_E |\nabla \xi(\cdot, t)|^2 \\
& +K \sum_{\gamma \in \Gamma_h} \frac{r^2}{h_\gamma} \|[\zeta(\cdot, t)]\|_\gamma^2 + K \int_0^t \sum_{E \in \mathcal{T}_h} \int_E |\nabla \xi|^2 + K \int_0^t \sum_{\gamma \in \Gamma_h} \frac{r^2}{h_\gamma} \left(\left\| \left[\frac{\partial \zeta}{\partial t} \right] \right\|_\gamma^2 + \left\| [\zeta] \right\|_\gamma^2 \right).
\end{aligned}$$

对 T_9 关于 t 积分并用分部积分, 有

$$\begin{aligned}
\left| \int_0^t T_9 \right| &= \left| \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_\gamma}{h_\gamma} \int_\gamma ([\zeta][\xi])(\cdot, t) - \int_0^t \sum_{\gamma \in \Gamma_h} \frac{r^2 \sigma_\gamma}{h_\gamma} \int_\gamma \left[\frac{\partial \zeta}{\partial t} \right] [\xi] \right| \\
&\leq \varepsilon J_0^\sigma(\xi(\cdot, t), \xi(\cdot, t)) + K \sum_{\gamma \in \Gamma_h} \frac{r^2}{h_\gamma} \|[\zeta(\cdot, t)]\|_\gamma^2 + K \int_0^t J_0^\sigma(\xi, \xi) \\
&\quad + K \int_0^t \sum_{\gamma \in \Gamma_h} \frac{r^2}{h_\gamma} \left(\|[\zeta]\|_\gamma^2 + \left\| \left[\frac{\partial \zeta}{\partial t} \right] \right\|_\gamma^2 \right), \\
T_2 + T_3 &= \frac{d}{dt} \sum_{\gamma \in \Gamma_h} \int_\gamma \{ (D(\mathbf{u}) - D(U)) \nabla c \cdot \mathbf{n} \} [\xi] - \sum_{\gamma \in \Gamma_h} \int_\gamma \left\{ (D(\mathbf{u}) - D(U)) \nabla \frac{\partial c}{\partial t} \cdot \mathbf{n} \right\} [\xi] \\
&\quad - \sum_{\gamma \in \Gamma_h} \int_\gamma \left\{ \frac{\partial}{\partial t} (D(\mathbf{u}) - D(U)) \nabla c \cdot \mathbf{n} \right\} [\xi] + \left(\frac{\partial}{\partial t} (D(\mathbf{u}) - D(U)) \nabla c \cdot \nabla \xi \right) \\
&\quad - \frac{d}{dt} \left((D(\mathbf{u}) - D(U)) \nabla c \cdot \nabla \xi \right) + \left((D(\mathbf{u}) - D(U)) \nabla \frac{\partial c}{\partial t} \cdot \nabla \xi \right) \equiv \sum_{i=1}^6 I_i.
\end{aligned}$$

注意到

$$\|\mathbf{u} - U\| \leq \|\boldsymbol{\rho}\| + \|\boldsymbol{\sigma}\|, \quad \left\| \frac{\partial(\mathbf{u} - U)}{\partial t} \right\| \leq \left\| \frac{\partial \boldsymbol{\rho}}{\partial t} \right\| + \left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|.$$

根据 $D(\mathbf{u})$ 的定义、 c 和 ∇c 的有界性、带 ε 的 Cauchy-Schwartz 不等式以及文献 [6] 中的迹不等式引理 3.1 和逆不等式引理 3.2, 可以得到

$$\begin{aligned}
\int_0^t |I_1| &\leq K \sum_{\gamma \in \Gamma_h} \int_\gamma \left(\left(\frac{r^2 \sigma_\gamma}{h_\gamma} \right)^{-1} \|(\mathbf{u} - U)(\cdot, t)\|_{(L^2(\gamma))^2}^2 + \frac{r^2 \sigma_\gamma}{h_\gamma} [\xi(\cdot, t)]^2 \right) \\
&\leq K \sigma_\gamma^{-1} (\|\boldsymbol{\rho}(\cdot, t)\|^2 + \|\boldsymbol{\sigma}(\cdot, t)\|^2) + \varepsilon J_0^\sigma(\xi(\cdot, t), \xi(\cdot, t)), \\
\int_0^t |I_2| &\leq K \int_0^t \left\| \nabla \frac{\partial c}{\partial t} \right\|_{L^\infty(\Omega)} \sum_{\gamma \in \Gamma_h} \|\mathbf{u} - U\|_{(L^2(\gamma))^2} \cdot \|[\xi]\| \\
&\leq K \sigma_\gamma^{-1} \int_0^t (\|\boldsymbol{\rho}\|^2 + \|\boldsymbol{\sigma}\|^2) + K \int_0^t J_0^\sigma(\xi, \xi), \\
\int_0^t |I_5| &\leq K \|\nabla c\|_{L^\infty(J; L^\infty(\Omega))} \cdot \|(\mathbf{u} - U)(\cdot, t)\|_{(L^2(\Omega))^2} \cdot \|\nabla \xi(\cdot, t)\| \\
&\leq K (\|\boldsymbol{\rho}(\cdot, t)\|^2 + \|\boldsymbol{\sigma}(\cdot, t)\|^2) + \varepsilon \|\nabla \xi(\cdot, t)\|^2, \\
\int_0^t |I_6| &\leq K \int_0^t \|\mathbf{u} - U\|_{(L^2(\Omega))^2} \cdot \left\| \nabla \frac{\partial c}{\partial t} \right\|_{\infty, \Omega} \cdot \|\nabla \xi\| \leq K \int_0^t (\|\boldsymbol{\rho}\|^2 + \|\boldsymbol{\sigma}\|^2) + K \int_0^t \|\nabla \xi\|^2.
\end{aligned}$$

根据嵌入定理和归纳假设 (11), 可得不等式

$$\left\| \frac{\partial \sigma}{\partial t} \right\|_{L^2(J; L^2(\Omega))} \leq K \left\| \frac{\partial \sigma}{\partial t} \right\|_{L^2(J; L^\infty(\Omega))} \leq M.$$

于是

$$\begin{aligned} \int_0^t |I_3| &\leq K \sum_{\gamma \in \Gamma_h} \left(\left\| \frac{\partial}{\partial t} (\mathbf{u} - \mathbf{U}) \right\|_{L^2(J; (L^2(\gamma))^2)} + \|(\mathbf{u} - \mathbf{U})(\cdot, t)\|_{(L^2(\gamma))^2} \|\mathbf{U}\|_{L^\infty(J; L^\infty(\gamma))} \right. \\ &\quad \cdot \left. \left\| \frac{\partial \mathbf{U}}{\partial t} \right\|_{L^2(J; L^\infty(\gamma))} \right) \|\nabla c\|_{L^\infty(J; L^\infty(\Omega))} \cdot \sum_{\gamma \in \Gamma_h} \|[\xi]\|_{L^2(J; L^2(\gamma))} \\ &\leq K \int_0^t J_0^\sigma(\xi, \xi) + K(\|\rho(\cdot, t)\|^2 + \|\sigma(\cdot, t)\|^2) + K \int_0^t \left\| \frac{\partial \rho}{\partial t} \right\|^2, \\ \int_0^t |I_4| &\leq K \left(\left\| \frac{\partial}{\partial t} (\mathbf{u} - \mathbf{U}) \right\|_{L^2(J; (L^2(\Omega))^2)} + \|(\mathbf{u} - \mathbf{U})(\cdot, t)\|_{(L^2(\Omega))^2} \|\mathbf{U}\|_{L^\infty(J; L^\infty(\Omega))} \right. \\ &\quad \cdot \left. \left\| \frac{\partial \mathbf{U}}{\partial t} \right\|_{L^2(J; L^\infty(\Omega))} \right) \|\nabla c\|_{L^\infty(J; L^\infty(\Omega))} \cdot \|\nabla \xi\|_{L^2(J; L^2(\Omega))} \\ &\leq K \int_0^t \|\nabla \xi\|^2 + K(\|\rho(\cdot, t)\|^2 + \|\sigma(\cdot, t)\|^2) + K \int_0^t \left\| \frac{\partial \rho}{\partial t} \right\|^2. \end{aligned}$$

因为 $\xi(x, 0) = 0$, 故

$$\int_0^t \sum_{E \in \mathcal{T}_h} \int_E \mathbf{D}(\mathbf{U}) \nabla \xi \cdot \nabla \frac{\partial \xi}{\partial t} = \frac{1}{2} \sum_{E \in \mathcal{T}_h} \int_E (\mathbf{D}(\mathbf{U}) \nabla \xi \cdot \nabla \xi)(t) - \int_0^t \sum_{E \in \mathcal{T}_h} \int_E \frac{\partial \mathbf{D}(\mathbf{U})}{\partial t} \nabla \xi \cdot \nabla \xi.$$

而

$$\begin{aligned} \sum_{E \in \mathcal{T}_h} \int_E (\mathbf{D}(\mathbf{U}) \nabla \xi \cdot \nabla \xi)(t) &\geq (d_m + \alpha_t |\mathbf{U}|) \sum_{E \in \mathcal{T}_h} \int_E |\nabla \xi(\cdot, t)|^2, \\ \left| \int_0^t \sum_{E \in \mathcal{T}_h} \int_E \frac{\partial \mathbf{D}(\mathbf{U})}{\partial t} \nabla \xi \cdot \nabla \xi \right| &\leq K \int_0^t \|\nabla \xi\|^2. \end{aligned}$$

设 σ_γ 足够大, 使得 $4K_1 \leq \min_{\gamma \in \Gamma_h} \sigma_\gamma$. 联立上述各估计式, 便得浓度方程的误差估计

$$\begin{aligned} &\int_0^t \left\| \frac{\partial \xi}{\partial t} \right\|^2 + \|\nabla \xi(\cdot, t)\|^2 + \|\xi(\cdot, t)\|^2 + J_0^\sigma(\xi(\cdot, t), \xi(\cdot, t)) \\ &\leq K \int_0^t \left(\|\rho\|^2 + \left\| \frac{\partial \rho}{\partial t} \right\|^2 + \|\sigma\|^2 + \|\zeta\|^2 + \|\xi\|^2 + \|\nabla \zeta\|^2 + \|\nabla \xi\|^2 + \left\| \frac{\partial \zeta}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \zeta}{\partial t} \right\|^2 \right) \\ &\quad + K \|\nabla \zeta(\cdot, t)\|^2 + K \|\rho(\cdot, t)\|^2 + K_2 \|\sigma(\cdot, t)\|^2 + K \int_0^t J_0^\sigma(\xi, \xi) \\ &\quad + K \int_0^t \sum_{\gamma \in \Gamma_h} \frac{h_\gamma}{r^2} \left(\|\nabla \zeta \cdot \mathbf{n}\|_\gamma^2 + \left\| \nabla \frac{\partial \zeta}{\partial t} \cdot \mathbf{n} \right\|_\gamma^2 \right) + K \int_0^t \sum_{\gamma \in \Gamma_h} \frac{r^2}{h_\gamma} \left(\|[\zeta]\|_\gamma^2 + \left\| \left[\frac{\partial \zeta}{\partial t} \right] \right\|_\gamma^2 \right) \\ &\quad + K \sum_{\gamma \in \Gamma_h} \left(\frac{h_\gamma}{r^2} \|\nabla \zeta(\cdot, t) \cdot \mathbf{n}\|_\gamma^2 + \frac{r^2}{h_\gamma} \|[\zeta(\cdot, t)]\|_\gamma^2 \right). \end{aligned} \quad (17)$$

接下来导出耦合问题的误差估计.

定理 1 令整数 λ, μ 和 ω 分别为函数 $c, \frac{\partial c}{\partial t}$ 和 p 的正则阶, λ_E, μ_E 和 ω_E 分别为它们在单元 E 上的值. 整数 r_E 表示单元 E 上间断有限元空间的阶, k 表示 RT_k 空间的阶. 假定

$$\min(r_E, \lambda_E - 1, \mu_E - 1, \omega_E - 1, k + 1) \geq 1, \quad \forall E \in \mathcal{T}_h. \quad (18)$$

设 (p, \mathbf{u}, c) 为问题 (1) 的解, 满足

$$p \in L^2(J; H^\omega(\mathcal{T}_h)), \quad c \in L^2(J; H^\lambda(\mathcal{T}_h)), \quad \frac{\partial c}{\partial t} \in L^2(J; H^\mu(\mathcal{T}_h)).$$

假设 $p, \nabla p, c$ 和 ∇c 本质有界, 记 (P, \mathbf{U}, C) 为耦合格式 (2)-(5) 的解. 那么

$$\begin{aligned} & \left\| \frac{\partial(c-C)}{\partial t} \right\|_{L^2(J; L^2(\Omega))} + \|c-C\|_{L^\infty(J; H^1(\Omega))} \\ & + (J_0^\sigma((c-C)(\cdot, t), (c-C)(\cdot, t)))^{\frac{1}{2}} + \|(\mathbf{u}-\mathbf{U})(\cdot, t)\| \\ & \leq K \sum_{E \in \mathcal{T}_h} \frac{h_E^{\min(k+1, \omega_E-1)}}{k^{\omega_E-1/2}} \|p\|_{\omega_E, E} \\ & + K \sum_{E \in \mathcal{T}_h} \left(\frac{h_E^{\min(r_E, \lambda_E-1)}}{r_E^{\lambda_E-3/2}} \|c\|_{\lambda_E, E} + \frac{h_E^{\min(r_E, \mu_E-1)}}{r_E^{\mu_E-3/2}} \left\| \frac{\partial c}{\partial t} \right\|_{\mu_E, E} \right). \end{aligned}$$

证明 注意到对任意的 $v, v(0) = 0$, 有

$$\|v(t)\|^2 = \int_0^t \frac{d}{dt} \|v(t)\|^2 = \int_0^t 2\|v(t)\| \cdot \left\| \frac{\partial v(t)}{\partial t} \right\| \leq \varepsilon \int_0^t \left\| \frac{\partial v}{\partial t} \right\|^2 + K \int_0^t \|v\|^2.$$

由于 $\xi(\cdot, 0) = 0$, 在上式中取 $v = \xi$, 将式 (14) 乘以 $K_2 + 1$, 关于时间积分后加到式 (17) 中, 并利用文献 [6] 中的迹不等式引理 3.1 和逆不等式引理 3.2、 hp 逼近性质、插值投影误差估计和 Gronwall 不等式, 得到

$$\begin{aligned} & \|\sigma(\cdot, t)\| + \int_0^t \left\| \frac{\partial \xi}{\partial t} \right\| + \|\nabla \xi(\cdot, t)\| + \|\xi(\cdot, t)\| + (J_0^\sigma(\xi(\cdot, t), \xi(\cdot, t)))^{1/2} \\ & \leq K \sum_{E \in \mathcal{T}_h} \frac{h_E^{\min(k+1, \omega_E-1)}}{k^{\omega_E-1/2}} \|p\|_{\omega_E, E} \\ & + K \sum_{E \in \mathcal{T}_h} \left(\frac{h_E^{\min(r_E, \lambda_E-1)}}{r_E^{\lambda_E-3/2}} \|c\|_{\lambda_E, E} + \frac{h_E^{\min(r_E, \mu_E-1)}}{r_E^{\mu_E-3/2}} \left\| \frac{\partial c}{\partial t} \right\|_{\mu_E, E} \right). \quad (19) \end{aligned}$$

再利用 ρ, η 和 ζ 的估计式以及三角不等式, 即可证得结论.

5 归纳假设的证明

接下来检验归纳假设 (10)-(11). 根据假设 (18) 和误差估计 (19), 并运用逆不等式 $\|v\|_{\infty, \Omega} \leq$

$Kh^{-1}\|\boldsymbol{v}\|$ ，对任意的 $\boldsymbol{v} \in V_k$ ，可知

$$\begin{aligned}\|\boldsymbol{\sigma}\|_{L^\infty(J;L^\infty(\Omega))} &\leq Kh^{-1}\|\boldsymbol{\sigma}\|_{L^\infty(J;L^2(\Omega))} \\ &\leq Kh^{-1} \sum_{E \in \mathcal{T}_h} \frac{h_E^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1/2}} \|p\|_{\omega_E,E} \\ &\quad + Kh^{-1} \left(\sum_{E \in \mathcal{T}_h} \left(\frac{h_E^{\min(r_E,\lambda_E-1)}}{r_E^{\lambda_E-3/2}} \|c\|_{\lambda_E,E} + \frac{h_E^{\min(r_E,\mu_E-1)}}{r_E^{\mu_E-3/2}} \left\| \frac{\partial c}{\partial t} \right\|_{\mu_E,E} \right) \right).\end{aligned}$$

可以选择 h 足够小使得 $\|\boldsymbol{\sigma}\|_{L^\infty(J;L^\infty(\Omega))} \leq L$ ，这就证得了式 (10)。为证明归纳假设 (11)，首先对式 (12) 和式 (13) 关于时间 t 求导并取 $w = \pi$ ， $\boldsymbol{v} = \frac{\partial \boldsymbol{\sigma}}{\partial t}$ ，得到

$$\begin{aligned}\left(\alpha(C) \frac{\partial \boldsymbol{\sigma}}{\partial t}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\right) &= \left(\frac{\partial((\alpha(C) - \alpha(c))\tilde{\boldsymbol{u}})}{\partial t}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\right) - \left(\frac{\partial \alpha(C)}{\partial C} \frac{\partial C}{\partial t} \boldsymbol{\sigma}, \frac{\partial \boldsymbol{\sigma}}{\partial t}\right) \equiv Y_1 + Y_2, \\ |Y_1| &= \left| \left(\tilde{\boldsymbol{u}} \left(\frac{\partial \alpha(C)}{\partial C} \frac{\partial C}{\partial t} - \frac{\partial \alpha(c)}{\partial c} \frac{\partial c}{\partial t} \right) + (\alpha(C) - \alpha(c)) \frac{\partial \tilde{\boldsymbol{u}}}{\partial t}, \frac{\partial \boldsymbol{\sigma}}{\partial t} \right) \right| \\ &\leq \varepsilon \left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|^2 + K \left(\left\| \frac{\partial \xi}{\partial t} \right\|^2 + \left\| \frac{\partial \zeta}{\partial t} \right\|^2 + \|\xi\|^2 + \|\zeta\|^2 \right).\end{aligned}$$

运用归纳假设 (10)，可以得出

$$\begin{aligned}|Y_2| &= \left| \left(\frac{\partial \alpha(C)}{\partial C} \left(\frac{\partial \tilde{c}}{\partial t} - \frac{\partial \xi}{\partial t} \right) \boldsymbol{\sigma}, \frac{\partial \boldsymbol{\sigma}}{\partial t} \right) \right| \\ &\leq K \left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\| \left(\|\boldsymbol{\sigma}\| + \|\boldsymbol{\sigma}\|_{\infty,\Omega} \cdot \left\| \frac{\partial \xi}{\partial t} \right\| \right) \leq \varepsilon \left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|^2 + K \|\boldsymbol{\sigma}\|^2 + K \left\| \frac{\partial \xi}{\partial t} \right\|^2.\end{aligned}$$

于是

$$\left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|^2 \leq K \left(\left\| \frac{\partial \xi}{\partial t} \right\|^2 + \left\| \frac{\partial \zeta}{\partial t} \right\|^2 + \|\xi\|^2 + \|\zeta\|^2 + \|\boldsymbol{\sigma}\|^2 \right) + 2\varepsilon \left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|^2.$$

注意到 $\boldsymbol{\sigma}(\cdot, 0) = 0$ ，用 Gronwall 引理和估计式 (19)，推出

$$\begin{aligned}\left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|_{L^2(J;L^\infty(\Omega))} &\leq Kh^{-1} \left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|_{L^2(J;L^2(\Omega))} \\ &\leq Kh^{-1} \sum_{E \in \mathcal{T}_h} \frac{h_E^{\min(k+1,\omega_E-1)}}{k^{\omega_E-1/2}} \|p\|_{\omega_E,E} \\ &\quad + Kh^{-1} \left(\sum_{E \in \mathcal{T}_h} \left(\frac{h_E^{\min(r_E,\lambda_E-1)}}{r_E^{\lambda_E-3/2}} \|c\|_{\lambda_E,E} + \frac{h_E^{\min(r_E,\mu_E-1)}}{r_E^{\mu_E-3/2}} \left\| \frac{\partial c}{\partial t} \right\|_{\mu_E,E} \right) \right).\end{aligned}$$

当 h 充分小时，可以得到

$$\left\| \frac{\partial \boldsymbol{\sigma}}{\partial t} \right\|_{L^2(J;L^\infty(\Omega))} \leq L.$$

参考文献:

[1] Douglas J, et al. The approximation of the pressure by a mixed method in the simulation of miscible displacement[J]. RAIRO Numerical Analysis, 1983, 17: 17-33

- [2] Douglas J, Roberts J E. Numerical methods for a model for compressible miscible displacement in porous media[J]. *Mathematics of Computation*, 1983, 41: 441-459
- [3] Chen Y, Yang J. A posteriori error estimation for a fully discrete discontinuous Galerkin approximation to a kind of singularly perturbed problems[J]. *Finite Elements in Analysis and Design*, 2007, 43(10): 757-770
- [4] Yang J, Chen Y. A unified a posteriori error analysis for discontinuous Galerkin approximations of reactive transport equations[J]. *Journal of Computational Mathematics*, 2006, 24(3): 425-434
- [5] Chen H, Chen Y. A combined mixed finite element and discontinuous Galerkin method for compressible miscible displacement problem[J]. *Journal of Xiangtan University (Natural Science)*, 2004, 26(2): 119-126
- [6] Cui M R. A combined mixed and discontinuous Galerkin method for compressible miscible displacement problem in porous media[J]. *Journal of Computational and Applied Mathematics*, 2007, 198(1): 19-34
- [7] Sun S, *et al.* A combined mixed finite element and discontinuous Galerkin method for miscible displacement problems in porous media[C]// *Proceedings of International Symposium on Computational and Applied PDEs*, Zhangjiajie, China, 2002: 321-348
- [8] Douglas J, Roberts J E. Global estimates for mixed finite element methods for second order elliptic equations[J]. *Mathematics of Computation*, 1985, 44: 39-52
- [9] Babuška I, Suri M. The optimal convergence rate of the p -version of the finite element method[J]. *SIAM Journal on Numerical Analysis*, 1987, 24: 750-776
- [10] Babuška I, Suri M. The hp version of the finite element method with quasiuniform meshes[J]. *RAIRO Mathematical Modelling and Numerical Analysis*, 1987, 21: 199-238

Error Analysis of a Combined Mixed Finite Element and Discontinuous Galerkin Method for Incompressible Miscible Displacement Problems

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Abstract: The numerical modeling of the miscible displacement in porous media is important and interesting in the oil recovery field, the environmental pollution problem and so on. A combined approximation is applied in this paper to the given miscible displacement problem including molecular diffusion and dispersion. The mixed finite element method is used for the pressure equation, and the concentration one is solved by the non-symmetric interior penalty discontinuous Galerkin method. Induction hypotheses are used to avoid the inconvenience of the cut-off operator. Based on the properties of the interpolation projection, an a priori hp error estimate is obtained.

Keywords: mixed finite element; discontinuous Galerkin; miscible displacement

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